# Real Numbers $\mathbb{R}$

## Rational Numbers $\mathbb{Q}$

Natural - $\{1, 2, 3, 4, 5, 6, 7, \ldots\}$

Whole - $\{0, 1, 2, 3, 4, \ldots\}$

Integers - $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Fractions - $\frac{a}{b}$, $b \neq 0$

Also includes:

All Terminating decimals - $0.5$, $0.75$

All Repeating Decimals - $0.333$, $0.7575\ldots$

Perfect Square Radicals - $\sqrt{25} = 5$ OR $\sqrt{144} = 12$

## Irrational Numbers $\bar{\mathbb{Q}}$

*<u>Cannot</u> be written as fractions

Also includes:

Infinite, Non-Repeating Decimals- $0.1569243\ldots$

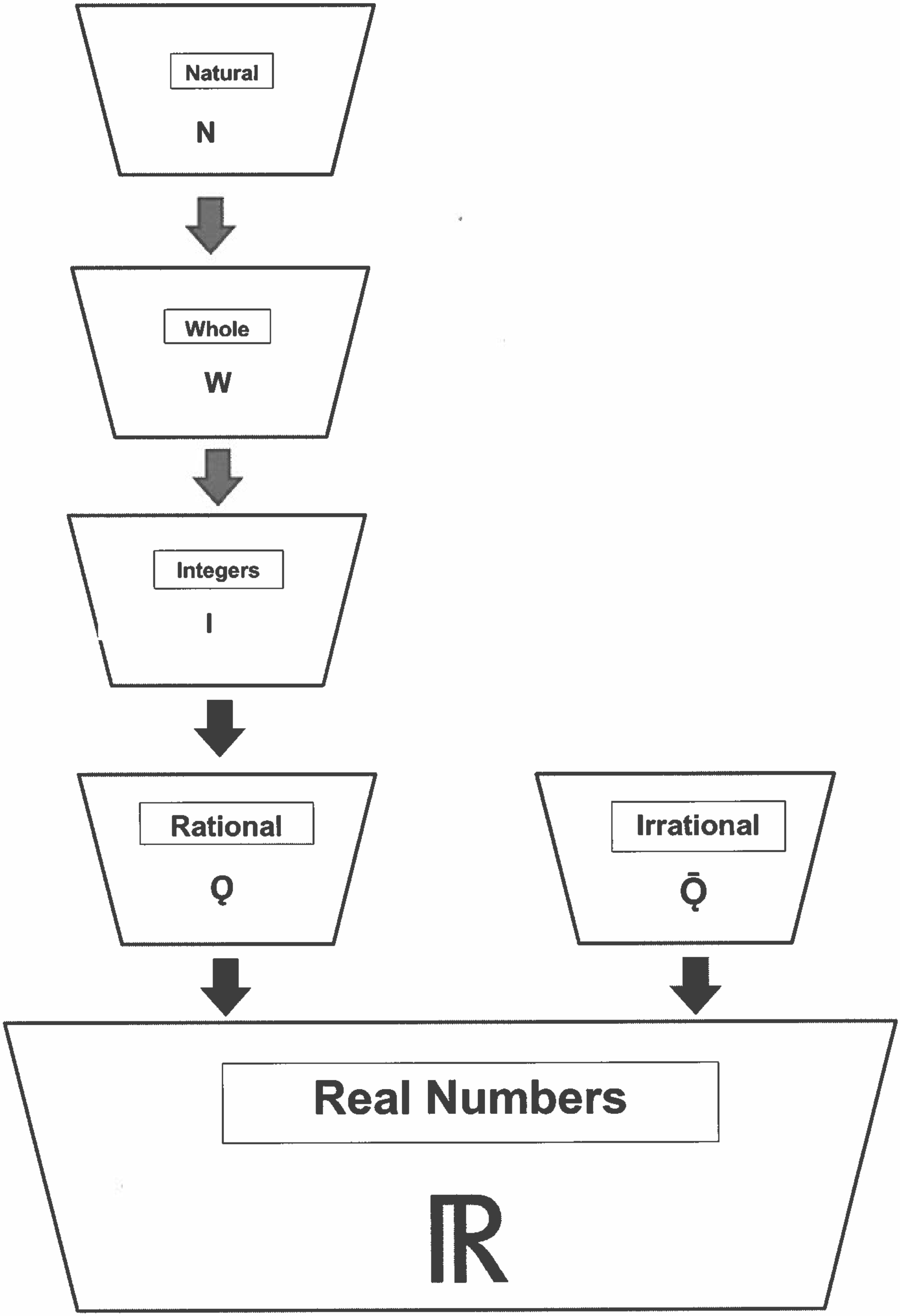
Non-Perfect Square Radicals – $\sqrt{5}$, $\sqrt{3}$, $\sqrt{7}$

Pattern Numbers – $0.02002000200002$

$\pi$ - Pi

Imaginary Numbers, $i$

$i = \sqrt{-1}$ (square roots of negative numbers)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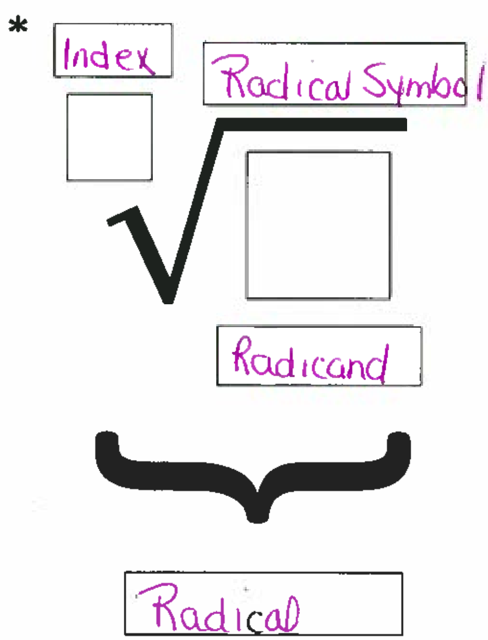
# Square and Cube Root Chart

| Number<br>$n$<br>root | Second Power<br>$n^2 = n x n$<br>Square of $n$ | Square Root<br>$\sqrt{\phantom{x}}$ perfect square | Third Power<br>$n^3 = n x n x n$<br>Cube of $n$ | Cubed Root<br>$\sqrt[3]{\phantom{x}}$ perfect square |
|---|---|---|---|---|
| 1 | $1^2$ | $\sqrt{1}$ | $1^3$ | $\sqrt[3]{1}$ |
| 2 | $2^2$ | $\sqrt{4}$ | $2^3$ | $\sqrt[3]{8}$ |
| 3 | $3^2$ | $\sqrt{9}$ | $3^3$ | $\sqrt[3]{27}$ |
| 4 | $4^2$ | $\sqrt{16}$ | $4^3$ | $\sqrt[3]{64}$ |
| 5 | $5^2$ | $\sqrt{25}$ | $5^3$ | $\sqrt[3]{125}$ |
| 6 | $6^2$ | $\sqrt{36}$ | $6^3$ | $\sqrt[3]{216}$ |
| 7 | $7^2$ | $\sqrt{49}$ | $7^3$ | $\sqrt[3]{343}$ |
| 8 | $8^2$ | $\sqrt{64}$ | $8^3$ | $\sqrt[3]{512}$ |
| 9 | $9^2$ | $\sqrt{81}$ | $9^3$ | $\sqrt[3]{729}$ |
| 10 | $10^2$ | $\sqrt{100}$ | $10^3$ | $\sqrt[3]{1000}$ |
| 11 | $11^2$ | $\sqrt{121}$ | $11^3$ | $\sqrt[3]{1331}$ |
| 12 | $12^2$ | $\sqrt{144}$ | $12^3$ | $\sqrt[3]{1728}$ |

## Parts of a Radical




*If the Index is not written, it is automatically a 2.

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Mixed and Entire Radicals Notes

Radicals are just numbers. As such, they can be added, subtracted, multiplied and divided. In this section, we will look at different ways to write a radical. We will use this fact to change radicals from one form to another, just like we can change fractions to decimals and back.

Equivalent expressions for any number have the same value. Equivalent means the same or equal.

$\sqrt{16 \cdot 9}$ is equivalent to $\sqrt{16} \cdot \sqrt{9}$ because

$\sqrt{144}$ $\qquad$ $4 \cdot 3$

$12$ $\qquad$ $12$

Similarly,

$\sqrt[3]{8 \bullet 27}$ is equivalent to $\sqrt[3]{8} \bullet \sqrt[3]{27}$ because

$\sqrt[3]{216}$ $\qquad$ $2 \cdot 3$

$6$ $\qquad$ $6$

Shown above is an example of the

**Multiplication Property of Radicals**

In general: $\sqrt[n]{ab} = \sqrt[n]{a} \bullet \sqrt[n]{b}$

where $n$ is a natural number, and $a$ and $b$ are real numbers

We can use this property to simplify square roots and cube roots that are not perfect squares or perfect cubes, but have factors that are perfect squares or perfect cubes.

**Example 1 – Simplifying Radicals Using Prime Factorization**

Simplify each radical using prime factors.

a) $\sqrt{63}$

$\sqrt{7} \cdot \sqrt{9}$

$3$

$3\sqrt{7}$

b) $\sqrt[3]{108}$

$\sqrt[3]{4} \cdot \sqrt[3]{27}$

$3$

$3\sqrt[3]{4}$

c) $\sqrt[4]{128}$

$\sqrt[4]{8} \cdot \sqrt[4]{16}$

$2$

$2\sqrt[4]{8}$

Another way of simplifying radicals is to use perfect square, cube, etc factors of the radicand. This is where that chart will really help. The next example will demonstrate:

$\sqrt{63}$

$7 \times 9$

$\boxed{3 \times 3}$

$3\sqrt{7}$

$\sqrt[3]{108}$

$2 \quad 54$

$2 \times 27$

$3 \quad 9$

$3 \quad 3$

$3\sqrt[3]{4}$

$\sqrt[4]{128}$

$2 \quad 64$

$2 \times 32$

$2 \times 16$

$2 \times 8$

$2 \times 4$

$2 \times 2$

$2\sqrt[4]{8}$

| x | $x^2$ | $x^3$ | $x^4$ | $x^5$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 4 | 8 | 16 | 32 |
| 3 | 9 | 27 | 81 | 243 |
| 4 | 16 | 64 | x | x |
| 5 | 25 | 125 | x | x |
| 6 | 36 | x | x | x |
| 7 | 49 | x | x | x |
| 8 | 64 | x | x | x |
| 9 | 81 | x | x | x |
| 10 | 100 | 1000 | x | x |
| 11 | 121 | x | x | x |
| 12 | 144 | x | x | x |
| 13 | 169 | x | x | x |
| 14 | 196 | x | x | x |
| 15 | 225 | x | x | x |

**Example 2 – Writing Radicals in Simplest Form**

Write each radical in simplest form, if possible.

a) $\sqrt{20}$

$\sqrt{4} \times \sqrt{5}$

$2\sqrt{5}$

b) $\sqrt{50}$

$\sqrt{25} \times \sqrt{2}$

$5\sqrt{2}$

c) $\sqrt{80}$

$\sqrt{16} \cdot \sqrt{5}$

$4\sqrt{5}$

d) $\sqrt{108}$

$\sqrt{36} \times \sqrt{3}$

$6\sqrt{3}$

e) $\sqrt{27}$

$\sqrt{9} \cdot \sqrt{3}$

$3\sqrt{3}$

f) $\sqrt[3]{32}$

$\sqrt[3]{8} \cdot \sqrt[3]{4}$

$2\sqrt[3]{4}$

g) $\sqrt[3]{135}$

$\sqrt[3]{27} \cdot \sqrt[3]{5}$

$3\sqrt[3]{5}$

h) $\sqrt[4]{48}$

$\sqrt[4]{16} \cdot \sqrt[4]{3}$

$2\sqrt[4]{3}$

Radicals of the form $\sqrt[n]{x}$ such as $\sqrt{80}$, $\sqrt[3]{144}$ and $\sqrt[4]{162}$ are called **entire radicals.**

Radicals of the form $a\sqrt[n]{x}$ such as $4\sqrt{5}$ , $2\sqrt[3]{18}$ , and $3\sqrt[4]{2}$ are called **mixed radicals.**

Entire radicals were rewritten as mixed radicals in *Examples 1* and *2*.

Any number can be written as the square root of its square; for example,

$2=\sqrt{2\cdot 2}$ , $3=\sqrt{3\cdot 3}$ , and $4=\sqrt{4\cdot 4}$, and so on.

Similarly, any number can be written as the cube root of its cube,

$5=\sqrt[3]{5\cdot 5\cdot 5}$

or the fourth root of its perfect fourth power.

$6=\sqrt[4]{6\cdot 6\cdot 6\cdot 6}$

We use this strategy to write a mixed radical as an entire radical.

**Example 3 - Writing Mixed Radicals as Entire Radicals**

Write each mixed radical as an entire radical

a) $7\sqrt{3}$

$7\sqrt{3}$

$\sqrt{3\times(7)^2}$

$\sqrt{3\times 49}$

$\sqrt{147}$

b) $2\sqrt[3]{4}$

$3\sqrt[3]{4}$

$\sqrt[3]{4\times(3)^3}$

$\sqrt[3]{108}$

c) $2\sqrt[5]{3}$

$2\sqrt[5]{3}$

$\sqrt[5]{3\times(2)^5}$

$\sqrt[5]{96}$

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**Fractional Exponents and Radicals Notes**

In grade 9, you learned that for powers with integral bases and whole numbers:

$$a^m . a^n = a^{m+n}$$

We can extend this law to powers with fractional exponents with numerator 1:

Consider the following:

$5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}}$ $\sqrt[2]{5^1} \times \sqrt[2]{5^1} = \sqrt[2]{25^1}$

$= 5^1$ $= 5$

$= 5$

From the above, we can see that $5^{\frac{1}{2}} = \sqrt{5}$

| **Powers with Rational Exponents with Numerator 1** |
|---|
| When $n$ is a natural number and $x$ is a rational number, $x^{\frac{1}{n}} = \sqrt[n]{x}$ |

**Example 1 – Evaluating Powers of the form $a^{\frac{1}{n}}$**

a) $1000^{\frac{1}{3}}$

$\sqrt[3]{1000^1}$

$10$

b) $0.25^{\frac{1}{2}}$

$\sqrt[2]{0.25^1}$

$0.5$

c) $(-8)^{\frac{1}{3}}$

$\sqrt[3]{-8^1}$

$-2$

d) $\left(\frac{16}{81}\right)^{\frac{1}{2}}$

$\dfrac{\sqrt[2]{16^1}}{\sqrt[2]{81^1}}$

$\dfrac{4}{9}$

A fraction can be written as a terminating or repeating decimal, so we can interpret powers with decimal exponents; for example, $0.2 = \frac{1}{5}$ so $32^{0.2} = 32^{\frac{1}{5}}$.

We can evaluate $32^{0.2}$ and $32^{\frac{1}{5}}$ on a calculator to show that both expressions have the same value.

$32^{0.2}$ $\quad 32^{\frac{1}{5}}$

$32^{\frac{2}{10}}$ $\quad \sqrt[5]{32}^{\,1}$

$32^{\frac{1}{5}}$ $\quad 2$

What about fractions where the numerator is not 1?

**Powers with Rational Exponents**

When $m$ and $n$ is a natural numbers and $x$ is a rational number

$$x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m \quad \text{or} \quad x^{\frac{m}{n}} = \sqrt[n]{x^m}$$

**Example 2 – Rewriting Powers in Radical and Exponent Form**

a) Write $26^{\frac{2}{5}}$ in both radical forms.

$\sqrt[5]{26}^{\,2}$

b) Write $\sqrt{6^5}$ in exponential form.

$6^{\frac{5}{1}}$

c) Write $\left(\sqrt[4]{19}\right)^3$ in exponential form.

$19^{\frac{3}{4}}$

**Example 3 – Evaluating Powers with Rational Exponents and Rational Bases.**

Evaluate

a) $0.01^{\frac{3}{2}}$ b) $(-27)^{\frac{4}{3}}$ c) $(81)^{\frac{3}{4}}$ d) $\left(\frac{8}{27}\right)^{\frac{2}{3}}$

a)
$(\sqrt[2]{0.01})^3$

$(0.1)^3$

$0.01$

b)
$(\sqrt[3]{-27})^4$

$(-3)^4$

$81$

c)
$(\sqrt[4]{81})^3$

$(3)^3$

$27$

d)
$\dfrac{(\sqrt[3]{8})^2}{(\sqrt[3]{27})^2}$

$\left(\frac{2}{3}\right)^2$

$\frac{4}{9}$

$-3^2 \rightarrow -3 \times 3 = -9$

$(-3)^2 \rightarrow -3 \times -3 = -9$

**Math 10C** **4.5 Negative Exponents**

1. Complete the below table using your calculator. If you get a decimal number, Math -> Frac it.

| Power | Answer |
|---|---|
| $4^3$ | 64 |
| $4^2$ | 16 |
| $4^1$ | 4 |
| $4^0$ | 1 |
| $4^{-1}$ | $\frac{1}{4}$ |
| $4^{-2}$ | $\frac{1}{16}$ |
| $4^{-3}$ | $\frac{1}{64}$ |

| Power | Answer |
|---|---|
| $9^3$ | 729 |
| $9^2$ | 81 |
| $9^1$ | 9 |
| $9^0$ | 1 |
| $9^{-1}$ | $\frac{1}{9}$ |
| $9^{-2}$ | $\frac{1}{81}$ |
| $9^{-3}$ | $\frac{1}{729}$ |

2. When the exponent went from positive numbers to negative numbers, did your answer change from positive to negative? No

3. How are the numbers $4^3$ and $4^{-3}$ related? Reciprocals

$64 \quad \frac{1}{64}$

4. If $5^3 = 125$, can you write the answer to $5^{-3}$ as a fraction without a calculator?

$\frac{1}{125}$

5. If $6^4 = 1296$, what is the value of $6^{-4}$ (as a fraction)? $\frac{1}{1296}$

6. Assume your calculator's negative button wasn't working. How would you use your calculator to determine the value of $7^{-3}$ as a fraction?

$7^3 \rightarrow 7 \times 7 \times 7 = 343 \quad 7^{-3} \rightarrow \frac{1}{343}$

7. Why are the answers to $8^{-3}$ and $8^{\frac{1}{3}}$ different?

$8^{-3} \rightarrow \frac{1}{512} \qquad 8^{\frac{1}{3}} \rightarrow \sqrt[3]{8^1} = 2$

$8^3 = 512 \qquad \frac{8}{1}^{-3} \rightarrow \frac{1}{8^3} = \frac{1}{512}$

reciprocal

$5^2 = 25 \qquad 5^{-2} \qquad 6^{-3}$

reciprocal $\frac{1}{5^2} \qquad \frac{1}{6^3}$

$\frac{1}{25} \qquad \frac{1}{216}$

# Negative Exponent & Reciprocal

Rule: $x^{-n} = \left(\frac{1}{x}\right)^n$

Reciprocal

Power Radical

$$\left(\frac{a}{b}\right)^{-\frac{n}{m}} = \left(\frac{b}{a}\right)^{\frac{n}{m}} \quad \sqrt[m]{\frac{b}{a}}^{\,n} \quad \frac{\sqrt[m]{b}^{\,n}}{\sqrt[m]{a}^{\,n}}$$

Reciprocal

$$3^2 = 9 \qquad 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

$$\left(\frac{-3}{4}\right)^{-3} = \left(-\frac{4}{3}\right)^{3} = -\frac{64}{27}$$

← odd (negative stays)

$$\left(-\frac{4}{3}\right)^{3} = \frac{(-4)^3}{(3)^3} = -\frac{64}{27}$$

$$8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{\boxed{\sqrt[3]{8}}^{\,2}} = \frac{1}{(2)^2} = \frac{1}{4}$$

$$\left(\frac{9}{16}\right)^{-\frac{3}{2}} = \left(\frac{16}{9}\right)^{\frac{3}{2}} = \frac{\boxed{\sqrt[2]{16}}^{\,3}}{\boxed{\sqrt[2]{9}}^{\,3}} = \frac{(4)^3}{(3)^3} = \frac{64}{27}$$

reciprocal

radical first

$$\frac{16}{1}^{-\frac{5}{4}} = \frac{1}{16^{\frac{5}{4}}} = \frac{1}{\boxed{\sqrt[4]{16}}^{\,5}} \quad \frac{1}{(2)^5} = \frac{1}{32}$$

$$\left(\frac{25}{36}\right)^{-\frac{1}{2}} = \left(\frac{36}{25}\right)^{\frac{1}{2}} = \frac{\sqrt{36}}{\sqrt{25}} = \frac{6}{5}$$

Radical

$$\left(-\frac{64}{125}\right)^{-\frac{5}{3}} = \left(-\frac{125}{64}\right)^{\frac{5}{3}} = -\frac{\boxed{\sqrt[3]{125}}^{\,5}}{\boxed{\sqrt[3]{64}}^{\,5}}$$

$$\sqrt[3]{6}^{\,5} \rightarrow 6^{\frac{5}{3}} \qquad \frac{(-5)^5}{(4)^5} = -\frac{3125}{1024}$$

Power

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Negative Exponent Notes

Rule: Negative Exponent means that the base is on the wrong side of the fraction line, so you need to flip the base to the other side (reciprocal).

$x^{-n} = \left(\frac{1}{x}\right)^n$ Eg. $4^{-2} = \frac{1}{(4)^2}$ $\left(\frac{3}{4}\right)^{-3} = \left(\frac{4}{3}\right)^3$

Negative Exponent $x^{\frac{-n}{m}} = \frac{1}{(x)^{\frac{n}{m}}}$ Written as a Radical $\frac{1}{(\sqrt[m]{x})^n}$

Eg. $16^{\frac{-1}{2}} = \left(\frac{1}{16}\right)^{\frac{1}{2}}$ $\frac{1}{(\sqrt[2]{16})}$

Negative Exponent $\left(\frac{a}{b}\right)^{\frac{-n}{m}} = \left(\frac{b}{a}\right)^{\frac{n}{m}}$ Written as a Radical $\left(\sqrt[m]{\frac{b}{a}}\right)^n$ or $\frac{(\sqrt[m]{b})^n}{(\sqrt[m]{a})^n}$

Eg. $\left(\frac{4}{25}\right)^{\frac{-3}{2}} = \left(\frac{25}{4}\right)^{\frac{3}{2}}$ $\left(\sqrt[2]{\frac{25}{4}}\right)^3$ $\frac{(\sqrt[2]{25})^3}{(\sqrt[2]{4})^3}$

| Rewrite with Rational Exponents | As a Radical | Evaluate |
|---|---|---|
| $36^{\frac{-1}{2}} = \frac{1}{(36)^{\frac{1}{2}}}$ | $\frac{1}{(\sqrt[2]{36})}$ | $\frac{1}{6}$ |
| $8^{\frac{-2}{3}} = \left(\frac{1}{8}\right)^{\frac{2}{3}}$ | $\frac{1}{(\sqrt[3]{8})^2}$ | $\frac{1}{(2)^2} = \frac{1}{4}$ |
| $\left(\frac{9}{16}\right)^{\frac{-3}{2}} = \left(\frac{16}{9}\right)^{\frac{3}{2}}$ | $\frac{(\sqrt[2]{16})^3}{(\sqrt[2]{9})^3}$ | $\frac{(4)^3}{(3)^3} = \frac{64}{27}$ |
| $\left(\frac{25}{36}\right)^{\frac{-1}{2}} = \left(\frac{36}{25}\right)^{\frac{1}{2}}$ | $\frac{(\sqrt[2]{36})^1}{(\sqrt[2]{25})^1}$ | $\frac{6}{5}$ |